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HYPERNORMAL CURVES ON THE GENERALIZED WEYL SPACE

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ABSTRACT

Hypernormal curves on Riemannian hypersurface have been defined and studied in the paper of U. P. Singh. In this paper, we define hypernormal curves on the generalized Weyl hypersurface and we determine tangential and normal components of the congruance of the hypernormal curves.
Keywords and Phrases: Hypernormal curves, Generalized weyl space, Intrinsic curvature vector, Normal curvature.
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GENELLEŞTİRİLMİŞ WEYL UZAYINDA HİPERNORMAL EĞRİLER

ÖZET

Riemann hiperyüzeylerinde hipernormal eğriler U. P. Singh'nin makalesinde tanımlanmış ve incelenmiştir. Bu çalışmada genelleştirilmiş Weyl uzayının hiperyüzeylerinde hipernormal eğriler tanımlanmış ve bu eğrilere ait kongrüansın teğetsel ve normal bileşenleri belirlenmiştir.
Anahtar Sözcükler: Hipernormal eğriler, Genelleştirilmiş weyl uzayı, Intrinsic eğrilik vektörü, Normal eğrilik.

1. INTRODUCTION

An n dimensional GW_n is said to be generalized Weyl space if it has an asymmetric conformal metric tensor g_{ij} and an asymmetric connection ∇_k satisfying the compatibility condition given by equation

$$\nabla_k g_{ij} = 2T_k g_{ij} \tag{1.1}$$

where T_k denotes a covariant vector field and ∇_k denotes the usual covariant derivative

Under a renormalization of the fundamental tensor of the form $\breve{g}_{ij} = \lambda^2 g_{ij}$ the covariant vector is transformed by the law
$\breve{T}_k = T_k + \partial_k \ln \lambda$, where λ is a scalar function on GW_n .

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Let L^i_{jk} denote the coefficients of the asymmetric connection ∇_k. So, a generalized Weyl space is shortly written as $GW_n\,(L^i_{jk}\,,\; g_{ij}\,,\; T_k\,)$.

The main properties of $GW_n\,(L^i_{jk}\,,\; g_{ij}\,,\; T_k\,)$ can be expressed as follows

$$g_{ij} = g_{(ij)} + g_{[ij]} \tag{1.2}$$

$$\nabla_k\, g_{(ij)} = 2\, g_{(ij)}\, T_k \tag{1.3}$$

$$\nabla_k\, g_{[ij]} = 2\, g_{[ij]}\, T_k \tag{1.4}$$

$$g_{(ik)} g^{(kl)} = \delta^l_i \tag{1.5}$$

$$\nabla_k g^{(ij)} = -2\, T_k\, g^{(ij)} \tag{1.6}$$

where $g_{(ij)}$ and $g_{[ij]}$ denote symmetric and antisymmetric part of g_{ij} respectively.

The symmetric part of connection coefficents L^i_{jk} are given as ([1], [2], [3])

$$L^i_{(jk)} = W^i_{jk} = \begin{bmatrix} i \\ jk \end{bmatrix} - (\delta^i_j\, T_k + \delta^i_k\, T_j - g_{jk} g^{mi} T_m) \tag{1.7}$$

where $\begin{bmatrix} i \\ jk \end{bmatrix}$ are second kind Christoffel symbols defined by

$$\begin{bmatrix} i \\ jk \end{bmatrix} = \frac{1}{2} g^{(ir)} [\frac{\partial g_{(jr)}}{\partial x^k} + \frac{\partial g_{(kr)}}{\partial x^j} - \frac{\partial g_{(jk)}}{\partial x^r}] \tag{1.8}$$

A quantity A is called a satelline of weight $\{p\}$ of tensor g_{ij}, if it admits a transformation of the form

$$\breve{A} = \lambda^p\, A \tag{1.9}$$

The prolonged covariant derivative of a satellite A of the tensor g_{ij} of weight $\{p\}$ is defined by

$$\dot{\nabla}_k\, A = \nabla_k A - p\, T_k A \tag{1.10}$$

Let $C:\; x^i = x^i(s)$ be curve in GW_n. The generalized covariant derivative along the curve C of the tensor field T is defined by

$$\frac{\dot{\delta} T}{\delta s} = \xi^k_{(1)}\, \dot{\nabla}_k\, T \tag{1.11}$$

where $\xi^k_{(1)}$ the components of the tangent vector of the curve C.

The Frenet equations of C may be written as [4]

$$\frac{\dot{\delta}\, \xi^i_{(\alpha)}}{\delta s} = \kappa_{(\alpha)} \xi^i_{(\alpha+1)} - \kappa_{(\alpha-1)} \xi^i_{(\alpha-1)} \quad (\alpha = 1, 2, ..., n\;\; ;\;\; \kappa_{(0)} = \kappa_{(n)} = 0\;) \tag{1.12}$$

In the above equation $\xi^i_{(\alpha)}$ $(\alpha = 2,...,n)$ denote the components of α - th normal with weight $\{-1\}$ normalized by the condition

$$g_{ij}\xi^i_{(\alpha)}\,\xi^j_{(\alpha)} = 1 \tag{1.13}$$

of the curve C and $\kappa_{(\alpha)}$ $(\alpha = 1,...,n-1)$ denote the α - th curvature of weight $\{-1\}$ of the curve C.

Let an $n-$ dimensional hypersurface GW_n given by the equations $y^\alpha = y^\alpha(x^i)$ ($\alpha = 1,..,n+1$; $i = 1,2,..,n$) be immersed in a generalized Weyl space GW_{n+1}.

The prolonged covariant derivative of the satellite A , relative to GW_{n+1} and GW_n are related by

$$\dot{\nabla}_k\, A = B^\gamma_k\, \dot{\nabla}_\gamma\, A \tag{1.14}$$

where $B^\gamma_k = \dfrac{\partial y^\gamma}{\partial x^k}$.

The components of any vector U relative to GW_{n+1} and GW_n are related by

$$U^\alpha = B^\alpha_i\, U^i \tag{1.15}$$

The prolonged covariant derivative B^α_i is given by

$$\dot{\nabla}_j\, B^\alpha_i = W_{ij}\, N^\alpha + A^h_{ij}\, B^\alpha_h \tag{1.16}$$

where W_{ij} are the components of second fundemental form of GW_n defined by

$$W_{ij} = g_{(\alpha\beta)} N^\beta\, \dot{\nabla}_j\, B^\alpha_i \tag{1.17}$$

and A^h_{ij} are defined by

$$A^h_{ij} = g_{(\alpha\beta)}\, \dot{\nabla}_j\, B^\alpha_i B^\beta_l g^{(hl)} \tag{1.18}$$

The components q^α and p^i of the first curvature vectors of the curve $C:\ x^i = x^i(s)$ with respect to GW_{n+1} and GW_n are given by [5]

$$q^\alpha = \frac{\dot{\delta}\,\varepsilon^\alpha_{(1)}}{\delta s} = \kappa_{(n)}\, N^\alpha + p^i B^\alpha_i + I^h B^\alpha_h \tag{1.19}$$

where $\kappa_{(n)}$ is the normal curvature of the hypersurface in the direction of the curve C defined by

$$\kappa_{(n)} = W_{ij}\,\frac{d\,x^i}{ds}\frac{d\,x^j}{ds} \tag{1.20}$$

and I^h are the components of intrinsic curvature vector of the curve C in the hypersurface defined by

$$I^h = A^h_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{1.21}$$

The prolonged covariant derivative of the unit normal is given by

$$\dot{\nabla}_i N^\alpha = -g^{(jk)} W_{ij} B^\alpha_k \tag{1.22}$$

2. HYPERNORMAL CURVES

We consider a congruence of curves given by unit vector field λ in GW_{n+1} as

$$\lambda^\alpha = r^i B^\alpha_i + C N^\alpha \tag{2.1}$$

If we use the relations

$$g_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1 \tag{2.2}$$

$$g_{\alpha\beta} N^\alpha N^\beta = 1 \tag{2.3}$$

$$g_{\alpha\beta} \lambda^\alpha N^\beta = \cos\theta \tag{2.4}$$

in (2.1) we have

$$C = \cos\theta \quad , \quad g_{ij} r^i r^j = \sin^2\theta \tag{2.5}$$

where θ denotes the angle between the unit vectors λ and N.

A curve of the hypersurface will be called a hypernormal curve if λ^α can be written as the following form

$$\lambda^\alpha = u\, q^\alpha + v \xi^\alpha_{(3)} \tag{2.6}$$

Using the relations (1.12), (1.16) and (1.19) we obtain

$$\begin{aligned} \frac{\delta^\bullet q^\alpha}{\delta s} &= -\kappa^2_{(1)} \xi^\alpha_{(1)} + \kappa_{(1)} \kappa_{(2)} \xi^\alpha_{(3)} + \frac{d \ln \kappa_{(1)}}{ds} q^\alpha \\ &= [\frac{\delta^\bullet [P^i + I^i]}{\delta s} - \kappa_n \xi^k_{(1)} g^{(ji)} W_{kj} + (p^r + I^r) A^i_{rk} \xi^k_{(1)}] B^\alpha_i + \\ &\quad + [\frac{d \ln \kappa_n}{ds} + (p^r + I^r) \xi^k_{(1)} W_{rk}] N^\alpha \end{aligned} \tag{2.7}$$

Eliminating $\xi^\alpha_{(3)}$ from (2.6) and (2.7) and using the equations (1.19), (2.1) and (2.5) we write

$$\begin{aligned} r^i &= u (p^i + I^i) + w [\kappa^2_{(1)} \xi^i_{(1)} - \frac{d \ln \kappa_{(1)}}{ds} (p^i + I^i) + \frac{\delta^\bullet [P^i + I^i]}{\delta s} - \\ &\quad + \kappa_n \xi^k_{(1)} g^{(ji)} W_{kj} + (p^r + I^r) A^i_{rk} \xi^k_{(1)}] \end{aligned} \tag{2.8}$$

$$c = \cos\theta = u \kappa_n + w [\frac{d \ln \kappa_n}{ds} + (p^r + I^r) \xi^k_{(1)} W_{rk} - \frac{d \ln \kappa_{(1)}}{ds} \kappa_n] \tag{2.9}$$

where w is defined by $w = \dfrac{v}{\kappa_{(1)}\,\kappa_{(2)}}$.

Using the relation (1.12) with respect to GW_n and considering the relation

$\kappa_{(1)}^2 = (k_{(1)} + \kappa_{(1)})^2 + \kappa_{(n)}^2$

we obtain

$$
\begin{aligned}
r^i &= u\,(p^i + I^i) + w[(\kappa_{(n)}^2 + 2k_{(1)}\kappa_{(I)} + \kappa_{(I)}^2)\xi_{(1)}^i + \\
&+ \frac{d\ln\kappa_{(1)}}{ds} I^i + \frac{d}{ds}(\ln\frac{\kappa_1}{k_1})p^i + \frac{\delta^{\bullet} I^i}{\delta s} + k_{(1)}k_{(2)}\xi_{(3)}^i + \\
&- \kappa_n\,\xi_{(1)}^k\,g^{(ji)}\,W_{kj} + (p^r + I^r)\,A_{rk}^i\,\xi_{(1)}^k\,]
\end{aligned}
\tag{2.10}
$$

where $k_{(1)}$, $k_{(2)}$ are first and second curvatures with respect to GW_n and $\kappa_{(I)}$ is scalar function defined by $\kappa_{(I)} = \sqrt{g_{ij} I^i I^j}$.

Thus we obtain that the tangential components r^i and the normal components c of any hypernormal curve in the hypersurface GW_n are determined by the equations (2.10) and (2.9) respectively.

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