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LOW RANK APPROXIMATE SOLUTIONS OBTAINED FROM A DIFFERENT APPLICATION OF GLOBAL ARNOLDİ METHOD

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ABSTRACT

The aim of this paper is to examine a numerical method for the computation of approximate solution of the continuous-time algebraic Riccati equation using Krylov subspace matrix. First of all, Global Arnoldi process is initiated to construct an orthonormal basis. In addition, Krylov subspace matrix is employed as projection method because it is one of the frequently referred method in the literature. Lastly, some numerical examples are given in order to explain how this method works.
Keywords: Algebraic Riccati equation, Krylov subspace, Arnoldi.

GLOBAL ARNOLDİ METODUNUN FARKLI BİR UYGULAMASINDAN ELDE EDİLEN DÜŞÜK RANKLI YAKLAŞIK ÇÖZÜMLER

ÖZET

Bu çalışmanın amacı Krylov alt uzay matrisi yardımıyla sürekli cebirsel Riccati denklemlerinin yaklaşık çözümünün hesaplanmasında kullanılan nümerik bir metodu incelemektir. İlk olarak ortanormal taban oluşturmak için global Arnoldi süreci başlatılmış, ayrıca literatürde sıklıkla başvurulan izdüşüm metotlarından biri olduğu için izdüşüm yöntemi olarak Krylov alt uzay matris kullanılmıştır. Son olarak, bu metodun nasıl çalıştığını açıklamak amacı ile bazı nümerik örnekler verilmiştir
Anahtar Sözcükler: Cebirsel Riccati denklemi, Krylov alt uzay, Arnoldi.

1. INTRODUCTION

Algebraic Riccati equations play a fundamental role in many areas, such as control theory, filter design, model reduction problems, differential equations and robust control [2, 3, 4, 5, 6, 7, 8]. Usually, in these applications, stable solution of continuous-time algebraic Riccati equation is desired. Such a solution X is symmetric positive semi-definite and each eigen value of the matrix $A-BB^TX$ has a negative real part. The stabilizing solution exists and it is unique under certain assumptions on the problem [6, 9, 17].

This paper presents the global Arnoldi method for the numerical solution of the continuous-time algebraic Riccati equation of the form

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$$A^T X + XA - XBB^T X + C^T C = 0 \quad (1)$$

where $A, X \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $C \in \mathcal{R}^{s\times n}$. The matrices B and C are assumed to be of full rank with $s << n$, $p << n$.

The remaining of the paper is organized as follows. In section 2, it has been composed matrix Krylov subspaces using the global Arnoldi process, also has been applied to compute low rank approximate solutions. In the last section, numerical experiments have been performed.

Throughout this paper, the following notations have been used. For X and Y two matrices in $\mathcal{R}^{n\times s}$ and the following inner product: $\langle X, Y\rangle_F = tr\left(X^T Y\right)$ where $tr(.)$ denotes trace. The associated norm is the Frobenius norm denoted by $\| \ \|_F$. $A \otimes B = \left[a_{i,j} B\right]$ is the Kronecker product of the matrices A and B. I_r is the identity of size $r \times r$ and $0_{r\times l}$ is the zero matrix of size $r \times l$.

2. LOW RANK APPROXIMATE SOLUTION

Before describing these global Arnoldi process, it is given some definitions and remarks on some matrix Krylov subspace methods [10, 11, 12, 13]

Let $A \in \mathcal{R}^{n\times n}$, $V \in \mathcal{R}^{n\times s}$ and m be a fixed integer. The matrix Krylov subspace $K_m(A, V) = span\left\{V, AV, ..., A^{m-1}V\right\}$ is the subspace spanned by the matrices $V, AV, ..., A^{m-1}V$. Note that $Z \in K_m(A, V)$ means that

$$Z = \sum_{i=0}^{m-1} \alpha_i A^i V, \ \alpha_i \in \mathcal{R}, \ i = 0, ..., m-1.$$

We recall that the previous subspace is different from the block Krylov subspace $K_m(A, V) = span\left\{V, AV, ..., A^{m-1}V\right\}$, where $Z \in K_m(A, V)$ means that

$$Z = \sum_{i=0}^{m-1} A^i V \Omega_i, \ \Omega_i \in \mathcal{R}^{s\times s}, \ i = 0, ..., m-1.$$

Also that block methods converge in less iterations than standard methods applied to each linear system, however block methods are very costly as m increases. Moreover, they can suffer from high memory requirements. Global methods are not demanded all these conditions. Therefore it is usually preferred in the applications [14, 15].

2.1. The Global Arnoldi Methods

Basically, the global Arnoldi algorithm is the standard Arnoldi algorithm when $s = 1$. Thus the global Arnoldi algorithm reduces to the standard Arnoldi algorithm [1]. The global Arnoldi algorithm constructs an F-orthonormal basis $V_1^a, V_2^a, \ldots, V_m^a$ of the matrix Krylov subspace $K_m(A,\ V)$; i.e.,

$V_i^a \perp_F V_j^a$ for $i \neq j$; $i, j = 1, \ldots, m$ and $\left\langle V_i^a, V_i^a \right\rangle_F = 1$ for $i = 1, \ldots, m$.

Algorithm 1. The modified global Arnoldi algorithm

A an $n \times n$ matrix, V an $n \times s$ matrix and m an integer.

1. $V_1^a = \dfrac{V}{\|V\|_F}$, (choose a vector V of norm 1)
2. For $j = 1, 2, \ldots, m$ Do:
3. Compute $\hat{V} := AV_j^a$;
4. For $i = 1, 2, \ldots, j$
5. $h_{ij} = \left\langle V_i^a, \hat{V} \right\rangle_F$;
6. $\hat{V} := \hat{V} - h_{ij} V_i^a$;
7. End Do
8. $h_{j+1,j} = \left\|\hat{V}\right\|_F$;
9. If $h_{j+1,j} = 0$ Stop
10. $V_{j+1}^a = \dfrac{\hat{V}}{h_{j+1,j}}$;
11. End Do.

Let $\mathcal{V}_m^a$ be the $n \times ms$ matrix $\mathcal{V}_m^a = \left[V_1^a, \ldots, V_m^a \right]$ and H_m be the $m \times m$ upper Hessenberg matrix whose nonzero entries $h_{i,j}$ are defined by algorithm1. $\hat{H}_m$ denotes the $(m+1) \times m$ upper Hessenberg matrix and H_m is the $m \times m$ matrix obtained from $\hat{H}_m$ by deleting its last row. If $e_m = (0, \ldots, 0, 1)^T \in \mathcal{R}^m$ and $E_m = (e_m \otimes I_s) = [0_s, \ldots, 0_s, I_s]^T$ then the following relations is satisfied [11]:

$$A\mathcal{V}_m^a = \mathcal{V}_m^a (H_m \otimes I_s) + h_{m+1,m} V_{m+1}^a E_m^T \tag{2}$$

and

$$A\mathcal{V}_m^a = \mathcal{V}_{m+1}^a \left(\hat{H}_m \otimes I_s \right). \tag{3}$$

In what follows, we will see how to extract low rank approximate solutions to the continuous-time algebraic Riccati equation (1). This will be done by projecting the initial problem onto the matrix Krylov subspaces $K_m\left(A^T, C^T\right)$ and $K_m\left(A, B\right)$. Then we solve the low dimensional CARE obtained and get an approximate solution to (1).

Theorem 1. Let l be the degree of the minimal polynomial of A^T for C^T and $\mathcal{V}_l^a = \left[V_1, \ldots, V_l\right]$ be the matrix obtained by applying the global Arnoldi algorithm to $\left(A^T, C^T\right)$ with $V_1 = \dfrac{C^T}{\|C\|_F}$. Let X_l denote the $n \times n$ matrix $X_l = \mathcal{V}_l^a Z_l \mathcal{V}_l^{a^T}$ where Z_l is a symmetric positive semi-definite solution of the following CARE [1]:

$$\left(H_l \otimes I_s\right) Z_l + Z_l \left(H_l^T \otimes I_s\right) - Z_l B_l B_l^T Z_l + C_l^T C_l = 0 \tag{4}$$

with $B_l = \mathcal{V}_l^{a^T} B$, $C_l = \|C\|_F \left(e_1^T \otimes I_s\right)$ and $e_1 = \left(1, 0, \ldots, 0\right)^T$ the first unit vector of $\mathcal{R}^l$.

Then X_l is a symmetric positive semi-definite solution of the CARE (1).

Repeat the same process as similar to $K_m\left(A, B\right)$.

The approximate solutions to (1) that we will consider have the following form:

$$X_l = \mathcal{V}_l^a Z_l \mathcal{V}_l^{a^T} .$$

Thus, CARE equation turns into

$$\left(H_l \otimes I_s\right) Z_l + Z_l \left(H_l^T \otimes I_s\right) - Z_l B_l B_l^T Z_l + C_l^T C_l = 0$$

in the low order equation.

2.2. The Coupled Algebraic Riccati Global Arnoldi Algorithm

A an $n \times n$ stable matrix, B an $n \times p$ matrix and C an $s \times n$ matrix.

1. Apply Algorithm 1 to the pair $\left(A, B\right)$.
2. Apply Algorithm 1 to the pair $\left(A^T, C^T\right)$.
3. The approximate solutions are represented as the matrix products:

$X_m^a = \mathcal{V}_m^a Z_m \mathcal{V}_m^{a^T}$ and $X_l = \mathcal{V}_l^a Z_l \mathcal{V}_l^{a^T}$.

3. NUMERICAL EXAMPLES

In this section, we present some numerical examples to illustrate the effectiveness of the global Arnoldi algorithm for continuous-time Riccati equations.

Example 1

$A=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}0\\ 1\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\end{bmatrix}$ matrices are computed for $K_m\left(A^T,C^T\right)$.

$H=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$, $V_l^a=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ matrices are substituted in the following equation.

$$\left(H_l\otimes I_s\right)Z_l+Z_l\left(H_l^T\otimes I_s\right)-Z_lB_lB_l^TZ_l+C_l^TC_l=0$$

As a result, Z_l , approximate solution is obtained.

$$Z_l=\begin{bmatrix}-3 & \dfrac{3+\sqrt{41}}{2}\\ \dfrac{3-\sqrt{41}}{2} & 3\end{bmatrix}.$$

Similarly, A,B,C matrices are computed for $K_m\left(A,B\right)$,

$H=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$, $V_l^a=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ then

$$\left(H_m\otimes I_s\right)Z_m+Z_m\left(H_m^{\ T}\otimes I_s\right)-Z_mB_mB_m^{\ T}Z_m+C_m^{\ T}C_m=0$$

As a result, Z_m , approximate solution is obtained

$$Z_m=\begin{bmatrix}3 & 1+3i\\ 1-3i & -3\end{bmatrix}.$$

When A,B,C matrices written in equation (1),

$$X=\begin{bmatrix}-3 & \dfrac{3+\sqrt{41}}{2}\\ \dfrac{3-\sqrt{41}}{2} & 3\end{bmatrix}$$

as can be seen here $Z_l=X$.

Example 2

$A=\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1\\ 0\end{bmatrix}$, $C=\begin{bmatrix}0 & 1\end{bmatrix}$ matrices are computed for $K_m\left(A^T,C^T\right)$.

$H=\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}$, $V_l^a=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ matrices are substituted in the following equation.

$$\left(H_l \otimes I_s\right)Z_l + Z_l\left(H_l^T \otimes I_s\right) - Z_l B_l B_l^T Z_l + C_l^T C_l = 0$$

As a result, Z_l, approximate solution is obtained.

$$Z_l = \begin{bmatrix} 4 & 3i \\ -3i & 2 \end{bmatrix}.$$

Similarly, A, B, C matrices are computed for $K_m\left(A, B\right)$,

$$H = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix},\ V_l^a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ then}$$

$$\left(H_m \otimes I_s\right)Z_m + Z_m\left(H_m^T \otimes I_s\right) - Z_m B_m B_m^T Z_m + C_m^T C_m = 0$$

As a result, Z_m, approximate solution is obtained.

$$Z_m = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & 1 \end{bmatrix}.$$

When A, B, C matrices written in equation (1),

$$X = \begin{bmatrix} 2 & 3i \\ -3i & 4 \end{bmatrix}$$

as can be seen here $X = \mathcal{V}_l^a Z_l \mathcal{V}_l^{a^T}$.

Example 3

$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ matrices are computed for $K_m\left(A^T, C^T\right)$.

$H = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$, $V_l^a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ matrices are substituted in the following equation.

$$\left(H_l \otimes I_s\right)Z_l + Z_l\left(H_l^T \otimes I_s\right) - Z_l B_l B_l^T Z_l + C_l^T C_l = 0$$

As a result, Z_l, approximate solution is obtained

$$Z_l = \begin{bmatrix} 2 & \frac{-1+\sqrt{11}i}{2} \\ \frac{-1-\sqrt{11}i}{2} & 1 \end{bmatrix}.$$

Similarly, A, B, C matrices computed for $K_m\left(A, B\right)$,

$H=\begin{bmatrix}1 & -2\\ 1 & 0\end{bmatrix}$, $V_l^a=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ then

$$\left(H_m\otimes I_s\right)Z_m+Z_m\left(H_m{}^T\otimes I_s\right)-Z_mB_mB_m{}^TZ_m+C_m{}^TC_m=0$$

As a result, Z_m, approximate solution is obtained.

$$Z_m=\begin{bmatrix}1 & \frac{1-\sqrt{3}i}{2}\\ \frac{1+\sqrt{3}i}{2} & 2\end{bmatrix}.$$

When A,B,C matrices written in equation (1),

$$X=\begin{bmatrix}1 & \frac{1+\sqrt{11}i}{2}\\ \frac{1-\sqrt{11}i}{2} & 2\end{bmatrix}$$

as can be seen here $X=\mathcal{V}_l^aZ_l\mathcal{V}_l^{a^T}$.

Example 4

$A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1\\ 0\end{bmatrix}$, $C=\begin{bmatrix}0 & 1\end{bmatrix}$ matrices are computed for $K_m\left(A^T,C^T\right)$.

$H=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $V=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ matrices are substituted in the following equation.

$$\left(H_l\otimes I_s\right)Z_l+Z_l\left(H_l^T\otimes I_s\right)-Z_lB_lB_l^TZ_l+C_l^TC_l=0$$

As a result, Z_l, approximate solution is obtained.

$$Z_l=\begin{bmatrix}2 & \sqrt{5}i\\ -\sqrt{5}i & 2\end{bmatrix}.$$

Similarly, A,B,C matrices are computed for $K_m\left(A,B\right)$,

$H=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, $V=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ then

$$\left(H_m\otimes I_s\right)Z_m+Z_m\left(H_m{}^T\otimes I_s\right)-Z_mB_mB_m{}^TZ_m+C_m{}^TC_m=0$$

As a result, Z_m, approximate solution is obtained.

$$Z_m = \begin{bmatrix} 2 & \dfrac{1+\sqrt{19}i}{2} \\ \dfrac{1-\sqrt{19}i}{2} & 2 \end{bmatrix}.$$

When A, B, C matrices written in equation (1),

$$X = \begin{bmatrix} 2 & \sqrt{5}i \\ -\sqrt{5}i & 2 \end{bmatrix}$$

as can be seen here $X = \mathcal{V}_l^a Z_l \mathcal{V}_l^{a^T}$.

4. CONCLUSION

In this study, it has been used the global Arnoldi method which is based on computing with low rank approximate solution of the continuous-time algebraic Riccati equation. It has been achieved two solution regarding both pair (A, B) and pair (A^T, C^T). Accordingly, the result obtained from pair (A, B) is quite different than that of exact solution. On the other hand, it has been found that approximate solution with low rank is the same with exact solution.

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