

FACTORIZATION PROPERTIES IN POLYNOMIAL EXTENSION OF UFR'S

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ABSTRACT

We investigate the factorization properties on the polynomial extension $A[X]$ of A where A is a UFR and show that $A[X]$ is a U-BFR for any UFR A . We also consider the ring structure $A + XI[X]$ where A is a UFR.

Keywords : Factorization, Polynomial rings.

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TÇA HALKALARIN POLİNOM GENİŞLEMELERİNDE ÇARPANLARA AYIRMA ÖZELLİKLERİ

ÖZET

A bir TÇA halka (tektürlü çarpanlara ayrılabilen halka) olmak üzere A 'nın polinom genişlemesi $A[X]$ üzerindeki çarpanlara ayırma özelliklerini araştırıyoruz ve herhangi bir TÇA halka A için $A[X]$ 'in U-KÇA halka (U-Kısıtlı Çarpanlarına Ayrılabilen Halka) olduğunu gösteriyoruz. A bir TÇA olmak üzere $A + XI[X]$ yapısındaki halkaları da göz önüne alıyoruz.

Anahtar Sözcükler : Çarpanlara ayırma, Polinom halkaları.

1. INTRODUCTION

One of the main problem in ring theory is to determine whether the polynomial extension of any commutative ring R possesses or not the properties belonging to R . The purpose of this paper is to investigate which factorization properties is exactly satisfied in the polynomial extension $R[X]$ of R where R is a UFR.

Let R be a commutative ring with identity. Any elements $a, b \in R$ are *associate*, denoted by $a \sim b$, if $a|b$ and $b|a$, that is, $(a) = (b)$. A nonunit $a \in R$ is an *irreducible* (or an *atom*) if $a = bc \Rightarrow a \sim b$ or $a \sim c$. Hence 0 is irreducible if and only if R is an integral domain. R is *atomic* if each nonzero nonunit element of R is a finite product of irreducible elements. A principal ideal ring (PIR) is called a *special principal ideal ring* (SPIR) if it has only one prime ideal $P \neq R$ and P is nilpotent, that is, $P^n = (0)$ for some integer $n > 0$. R is said to

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be *presimplifiable* if $x = xy \Rightarrow x = 0$ or $y \in U(R)$. SPIR's and integral domains are examples of presimplifiable rings.

Classical irreducible decomposition may cause the bad behavior of factorization for some elements in a commutative ring because of nontrivial idempotents. For example $5 = 5^n$ for every positive integer n in \mathbb{Z}_{20} . A U-decomposition which is introduced by Fletcher [9] eliminates this bad behavior of a factorization. A *U-decomposition* of $r \in R$ is a factorization

$$r = (p'_1 \dots p'_k)(p_1 \dots p_n)$$

such that

- (i) the p'_i 's and p_j 's are irreducible
- (ii) $p'_i(p_1 \dots p_n) = (p_1 \dots p_n)$ for $i = 1, \dots, k$ and
- (iii) $p_j(p_1 \dots p_{j-1} p_{j+1} \dots p_n) \neq (p_1 \dots p_n)$ for $j = 1, \dots, n$.

Let $r = (p'_1 \dots p'_k)(p_1 \dots p_n)$ be a U-decomposition of $r \in R$. Then the product $p_1 \dots p_n$ is called the *relevant part* and the other is the *irrelevant part*. Any irreducible decomposition can be rearranged to a U-decomposition. Two U-decompositions

$$r = (p'_1 \dots p'_k)(p_1 \dots p_n) = (q'_1 \dots q'_l)(q_1 \dots q_m)$$

are *associates* if (i) $n = m$ and (ii) p_i and q_i are associates for $i = 1, \dots, n$, after a suitable change in the order of the factors in the relevant parts. R is called *unique factorization ring* (UFR) if each nonunit has a U-decomposition, that is, R is atomic and any two U-decompositions of a nonunit element of R are associates. Fletcher shows that R is a UFR if and only if R is a finite direct product of UFD's and SPIR's [10].

There are a family of factorization properties which are weaker than unique factorization. For a detailed study of these factorization properties in integral domain see Refs. [4,5,6]. In this paper we consider these factorization properties for a commutative ring with zero divisors. Any commutative ring R is called a *bounded factorization ring* (BFR) if for each nonzero nonunit $a \in R$, there exist a natural number $N(a)$ so that for any factorization $a = a_1 \dots a_n$ of a where each a_i is a nonunit we have $n < N(a)$. If we replace the condition that p'_i 's and p_j 's are irreducible by p'_i 's and p_j 's are nonunits in the definition of U-decomposition, we have the definition of *U-factorization*. A commutative ring R is a U-BFR if for each nonzero nonunit $a \in R$, there exist a natural number $N(a)$ so that for each U-factorization of a , $a = (a_1, \dots, a_n)(b_1, \dots, b_m)$, $m < N(a)$ [1]. R is a UFR $\Rightarrow R$ is a U-BFR. R is called a *U-half-factorial ring* (U-HFR) if R is U-atomic (that is, every nonzero nonunit has a U-factorization in which all the relevant divisors are irreducible) and if $a = (a_1, \dots, a_n)(b_1, \dots, b_m) = (c_1, \dots, c_l)(d_1, \dots, d_s)$ are two U-factorizations with b_i, d_j irreducible, then $m = s$. It is clear that UFR \Rightarrow U-HFR. For these factorization properties see [1,7,8].

For an integral domain D it is well known that D is a UFD $\Leftrightarrow D[X]$ is a UFD. But if zero divisors are present the situation is not so clear. In [2] Anderson and Markanda show that $R[X]$ is a UFR if and only if R is a finite direct product of UFD's. In other words $R[X]$ is not a UFR if R is a SPIR. In fact R is not even a U-HFR if R is a SPIR. For example the element $2X^2$ has two distinct irreducible decomposition (and U-decomposition) in \mathbb{Z}_4 which is a UFR, namely

$$2X^2 = 2(2 + X^2) = 2.X.X$$

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In here we continue to investigate these factorization properties in $R[X]$ where R is a UFR and give a positive result for the concepts a U-BFR and a BFR. More precisely we show that if R is a SPIR then $R[X]$ is a BFR and if R is a UFR then $R[X]$ is a U-BFR. We also consider these factorization properties in the rings $R + XI[X]$ for any ideal I of R where R is a UFR.

In [12] Gonzalez, Pelerin and Robert show that $A + XI[X]$ is a HFD (half-factorial domain) if and only if I is a prime ideal of A for the domain case where A is a UFD. In here we investigate which factorization properties are satisfied in the rings $A + XI[X]$ for any ideal I of A where A is a UFR and show that $A + XI[X]$ is always a U-BFR for any ideal I of A . For any undefined terminology or notations, see [11].

2. BOUNDED FACTORIZATION PROPERTIES ON $A[X]$

Suppose A is a SPIR and $P = (p)$ is the unique prime ideal of A , where $P^n = (0)$ and n is the smallest integer which satisfies $P^n = (0)$. If $n = 1$ then A is a field. So from now on, we assume that $n > 1$ and A, P and n will be as above unless otherwise stated.

Proposition 2.1: Let A be a SPIR. If $f(X) = a_0 + a_1X + \dots + a_mX^m \in A[X]$ is any irreducible element in $A[X]$ then $f(X)$ is one of the following forms up to associate:

- (i) $f(X) = p$
- (ii) $f(X) = X$
- (iii) $f(X) = a_0 + a_1X + \dots + a_{k-1}X^{k-1} + X^k + a_{k+1}X^{k+1} + \dots + a_mX^m$ for some $1 \leq k \leq m$ where $a_0, a_{k+1}, \dots, a_m \in P$
- (iv) $f(X) = 1 + a_1X + \dots + a_{k-1}X^{k-1} + X^k + a_{k+1}X^{k+1} + \dots + a_mX^m$ for some $1 \leq k \leq m$ where $a_{k+1}, \dots, a_m \in P$.

Proof : Suppose $a_i \in P$ for all $0 \leq i \leq m$. We can write $a_i = pa'_i \quad \exists a'_i \in A$. Then $f(X) = p(a'_0 + a'_1X + \dots + a'_mX^m)$. Since $f(X)$ is irreducible in $A[X]$, $f(X) \sim p$ or $f(X) \sim f'(X)$ where $f'(X) = a'_0 + a'_1X + \dots + a'_mX^m$.

$$\begin{aligned} f(X) \sim p &\Rightarrow p = f(X)c(X) \quad \exists c(X) \in A[X] \\ &\Rightarrow f(X) = f(X)c(X)f'(X). \end{aligned}$$

Since $A[X]$ is presimplifiable and $0 \neq f(X)$, $c(X)f'(X)$ is a unit in $A[X]$, and hence $f'(X)$ is a unit in $A[X]$. So we may take $f(X) = p$ up to associate. If $f(X) \sim f'(X)$ then

$$\begin{aligned} f'(X) = f(X)d(X) \quad \exists d(X) \in A[X] &\Rightarrow f(X) = p.f(X)d(X) \\ &\Rightarrow p.d(X) \in U(A). \end{aligned}$$

But this is a contradiction since p is not a unit. Therefore if all $a_i \in P$ then $f(X) = p$ up to unit. Now suppose $f(X) \notin P[X]$. Consider the constant term of $f(X)$:

Case I: Suppose $a_0 = 0$. Then $f(X) = X.f'(X) \quad \exists f'(X) \in A[X]$. So either $f(X) \sim X$ or $f(X) \sim f'(X)$. $f(X) \sim f'(X)$ gives X is a unit as above which is a contradiction. So $f(X) \sim X$ and $f'(X)$ is a unit in $A[X]$. Hence we take $f(X) = X$ up to associate. Now $a_0 \neq 0$. Then either $a_0 \in P$ or $a_0 \in U(A)$.

Case II: Let $a_0 \in P$. Then for some $1 \leq k \leq m$ $a_k \notin P$ since $f(X) \notin P[X]$. So $a_k \notin P \Rightarrow a_k \in U(A)$. So we may take $a_k = 1$.

Case III: Let $a_0 \in U(A)$. We know that any element of $A[X]$ is a unit in $A[X]$ if and only if the constant term is a unit in A and all other coefficients are nilpotents in A . Any nonunit element of A is a nilpotent. So a_k must be a unit in A for some $1 \leq k \leq m$ since $f(X)$ is not a unit in $A[X]$. So again we may take $a_k = 1$.

Proposition 2.2: If A is a SPIR then $A[X]$ is a BFR.

Proof: Let $f(X) = a_0 + a_1X + \dots + a_nX^n \in A[X]$ be a nonzero nonunit element of $A[X]$. First note that $A[X]$ is presimplifiable since 0 is primary in A [3]. Collecting all the factor p in each coefficient a_i , we can write uniquely each a_i as $a_i = p^{\alpha_i}u_i$ where $0 \leq \alpha_i \leq n$ and $u_i \in U(A)$. Then $f(X) = p^{\alpha_0}u_0 + p^{\alpha_1}u_1X + \dots + p^{\alpha_n}u_nX^n \in A[X]$. Since $f(X) \neq 0$ the number of irreducible factor p in any irreducible decomposition of $f(X)$ is at most $n-1$. Now we show that in any factorization of $f(X)$ into a product of irreducible elements the number of irreducible factors as in Proposition 2.1 is at most $\deg f(X) = s$. It is obvious that the number of irreducible factor X 's are at most s . Let $f(X) = p \dots pX \dots Xf_1 \dots f_t$ be an irreducible decomposition of $f(X)$ where f_i is irreducible as in Proposition 2.1 other than p and X . Let

$$f_1 = a_0 + \dots + a_{k_1-1}X^{k_1-1} + X^{k_1} + a_{k_1+1}X^{k_1+1} + \dots + a_tX^t,$$

$$f_2 = b_0 + \dots + b_{k_2-1}X^{k_2-1} + X^{k_2} + a_{k_2+1}X^{k_2+1} + \dots + a_mX^m$$

where a_{k_1} and b_{k_2} are the last coefficients of f_1 and f_2 which is equal to 1, respectively.

Consider the coefficient of $X^{k_1+k_2}$ of the product f_1f_2 :

$$\sum_{j=0}^{k_1+k_2} a_j b_{k_1+k_2-j}.$$

$$\text{If } j > k_1 \Rightarrow a_j \in P \Rightarrow a_j b_{k_1+k_2-j} \in P,$$

$$\text{if } j < k_1 \Rightarrow b_{k_1+k_2-j} \in P \Rightarrow a_j b_{k_1+k_2-j} \in P,$$

$$\text{if } j = k_1 \Rightarrow a_j = b_{k_1+k_2-j} = 1 \Rightarrow a_j b_{k_1+k_2-j} = 1.$$

So

$$\sum_{j=0}^{k_1+k_2} a_j b_{k_1+k_2-j} = 1 + pr, \exists r \in A.$$

Since $1 + pr \notin P$, $1 + pr$ is a unit in A , and hence not a zero element. Hence by induction on t we can see that the coefficient of $X^{k_1+\dots+k_t}$ in the product $f_1 \dots f_t$ is not zero where $k_i \geq 1$. So if $t > s$ then $\deg(f(X)) < \deg(p \dots pX \dots Xf_1 \dots f_t)$ which is a contradiction. Thus $t \leq s$ and hence $A[X]$ is a BFR.

Following theorem is in [1]. For a commutative ring R , $a \in R$ is called U-bounded if $\sup\{m \mid a = (a_1, \dots, a_n)(b_1, \dots, b_m) \text{ is a U-factorization of } a\} < \infty$.

Theorem 2.3: Let R_1, R_2, \dots, R_n be commutative rings, $n > 1$, and let $R = R_1 \times \dots \times R_n$. Then R is a U-BFR \Leftrightarrow each R_i is a U-BFR and 0_{R_i} is U-bounded. Hence 0_R is U-bounded.

Now we can state the main result of this paper as a corollary.

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Corollary 2.4: If A is a UFR then $A[X]$ is a U-BFR.

Proof : Since A is a UFR, A is a finite direct product of UFD's and SPIR's, say $A = A_1 \times \dots \times A_n$. Then $A[X] = A_1[X] \times \dots \times A_n[X]$. If A_i is a UFD then clearly $A_i[X]$ is UFD and hence a U-BFR. If A_i is a SPIR then $A_i[X]$ is a U-BFR by Proposition 2.2. Hence by Theorem 2.3, $A[X]$ is U-BFR.

Lemma 2.5: Let $A \subset B$ be an extension of commutative rings. If B is a BFR and $U(B) \cap A = U(A)$ then A is also a BFR.

Proof: Let $a \in A$ be a nonzero nonunit and let $a = a_1 \dots a_n$ be any factorization of a into nonunits. Then $a_i \notin U(B)$ for $i=1, \dots, n$. Hence $a = a_1 \dots a_n$ is a factorization of a in B . Since B is a BFR, $n \leq N(a)$ for some positive integer $N(a)$. So A is a BFR.

Proposition 2.6: If A is a UFR then $A + XI[X]$ is a U-BFR for any ideal I of A .

Proof : Let $A = A_1 \times \dots \times A_n$, a finite direct product of UFD's and SPIR's. Then I is of the form $I = I_1 \times \dots \times I_n$ where each I_i is an ideal of A_i . So

$$\begin{aligned} A + XI[X] &= (A_1 \times \dots \times A_n) + X(I_1 \times \dots \times I_n)[X] \\ &= (A_1 + XI_1[X]) \times \dots \times (A_n + XI_n[X]) \end{aligned}$$

If A_i is a UFD then by Lemma 2.5 $A_i + XI_i[X]$ is a BFR since $A_i + XI_i[X] \subseteq A_i[X]$ and $U(A_i + XI_i[X]) = U(A_i[X]) = U(A_i)$. If A_i is a SPIR then $A_i + XI_i[X]$ is again a BFR since $A_i[X]$ is a BFR by Proposition 2.2 and $U(A_i[X]) \cap (A_i + XI_i[X]) = U(A_i + XI_i[X])$. Clearly 0_{A_i} is always U-bounded in each case. Hence $A + XI[X]$ is a U-BFR for every ideal I of A by Theorem 2.4.

3. RESULTS AND DISCUSSION

In this paper we concentrate on polynomial extension of a UFR with zero divisors to investigate factorization properties related to it. We show that if A is a UFR then $A[X]$ is always a U-BFR and $A + XI[X]$ is always a U-BFR for any ideal I of A . We do not know these results are remain valid if the term "U-BFR" is substituted for "UFR".

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